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A Canonical Constant Elasticity of Substitution (CES) Production Function*

Ki-Hong Choi† Sungwhee Shin‡

Abstract We propose a canonical form of the CES production function. It includes the normalized CES function as a special case. It can represent factor-augmenting technological progresses through the factor efficiency term. In general, it can represent a family of CES-type function indexed by a parameter. The parameter may represent time, nations, regions, or industries. Finally, it treats well the concern on dimensional homogeneity.

Keywords Constant Elasticity of Substitution, Factor-augmenting Technological Progress, Dimensional Homogeneity

JEL Classification C68, D24

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†Korea Institute of Public Finance, 336 Sichungdae-ro, Sejong Special Autonomous City 30147, Republic of Korea. Email: khchoi@kipf.re.kr.

‡Corresponding author. School of Economics, University of Seoul, 163 Seoulsiripdae-ro, Dongdaemun-gu, Seoul 02504, Republic of Korea. Email: sungshin@uos.ac.kr

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1. INTRODUCTION

The Cobb-Douglas production function has been used widely in macroeconomics. This is due to not only analytical tractability but also the Kaldor's stylized fact that factor income shares has been constant in light of the long-term historical trend. The Cobb-Douglas production function is consistent with the balanced growth and generates constant labor share. Recently, Piketty (2014) claims that labor income share has decreased during the period of 1970-2010. This raised interest in the constant elasticity of substitution (CES) production function. The CES production function is consistent with balanced growth and the constant factor shares in the balanced growth path when the technological progress is labor-augmenting. Candore and Levine (2012) solved for the balanced growth path in a real business cycle model with CES production function which is equivalent to that used in this paper. The CES production function also admits the possibility of decreasing labor share in the transition period to a steady state. The CES function is widely used not only as production functions but also as utility functions in computable general equilibrium models. (Kim and Shin, 2013)

Arrow *et al.* (1961) proposed the original CES production function as a production function. It is widely used as a typical production function in computable general equilibrium models. For instance, Auerbach and Kotlikoff (1987) use the CES function as a production function.

However, the original CES production function has some limits. First, the original form is awkward at representing factor-augmenting technological progress in the context of dynamic economic growth analysis. Second, it invokes the concern on dimensional homogeneity. (De Jong and Kumar, 1972) For instance, the summation of capital and labor is not possible if the unit of measurement is not unified.

We propose a form of CES function. It contains the normalized (or calibrated share form of) CES function (Rutherford, 1998; Klump *et al.*, 2012) as a special case. It also can represent factor-augmenting technological progresses as well. Furthermore, It calms the concern on dimensional homogeneity.

2. A CANONICAL CES PRODUCTION FUNCTION

The original CES production function proposed by Arrow *et al.* (1961) is

$$y = f(x_1, x_2) = A[a_1 x_1^{\rho} + a_2 x_2^{\rho}]^{\frac{1}{\rho}} \tag{1}$$

where $A > 0, a_1 \geq 0, a_2 \geq 0, a_1 + a_2 = 1$ and $\rho < 1$.

The parameter A is an efficiency or scale parameter, parameters a_1, a_2 are cost share parameters or distribution parameters, and ρ is the substitution parameter. The efficiency or scale parameter A cannot represent factor-specific efficiency. Thus, the CES function cannot represent specific factor-augmenting technological progresses. Note that when the elasticity of substitution approaches zero, the CES function converges to the Leontief function, where the isoquant has kinks at the 45-degree line: $f(x_1,x_2) = \min A[x_1,x_2]$.

On the other hand, La Grandville (2017) derived the normalized CES function from the relation between per capita income and wage rate, using the base period value of income and factor prices: $y = \bar{y}[\sum_{i=1}^{2} \bar{\theta}_i (\frac{x_i}{\bar{x}_i})^\rho]^{\frac{1}{\rho}}$ where $\bar{\theta}_i = \frac{\bar{w}_i \bar{x}_i}{\sum_i \bar{w}_i \bar{x}_i}$ and the barred variables denote the base period values of the variables.

This implies that $a_i = \frac{\bar{w}_i \bar{x}_i{}^{1-\rho}}{\sum_j \bar{w}_j \bar{x}_j{}^{1-\rho}}$ and $A = \bar{y}[\frac{\sum_i \bar{w}_i \bar{x}_i{}^{1-\rho}}{\sum_j \bar{w}_j \bar{x}_j}]^{1/\rho}$ in the original form of Arrow *et al.* (1961). Thus the parameters a_i and A are dependent upon the elasticity of substitution. This provides the solution for the puzzle why the original form of CES function converges to the Leontief function, where the isoquant has kinks at the 45-degree line as $\sigma \to 0$. When taking the limit where $\sigma \to 0$, we treated the parameters as fixed constants. This was the source of the problem.

The normalized CES form is attractive in that it converges to the Leontief function where the isoquant has kinks at the lines going through $(\bar{x_1}, \bar{x_2})$ from the origin as $\sigma \to 0$. It also makes the calibration of parameters easier. Moreover, it has the unit cost (price) function which is genuinely symmetric to the production function.

We propose the following form for the CES function.

Definition 1: A canonical CES production function is

$$f(x_1,x_2) = A[a_1(b_1x_1)^\rho + a_2(b_2x_2)^\rho]^{\frac{1}{\rho}} \qquad or \tag{2a}$$

$$f(x_1,x_2) = A[a_1(b_1x_1)^{1-\frac{1}{\sigma}} + a_2(b_2x_2)^{1-\frac{1}{\sigma}}]^{\frac{1}{1-\frac{1}{\sigma}}} \tag{2b}$$

where $A > 0$ is an efficiency or scale parameter, $a_1 \geq 0, a_2 \geq 0, a_1 + a_2 = 1$ are the cost share or distribution parameters, $b_1, b_2 \geq 0$ are the factors' efficiency or scale parameters, and $\sigma = 1/(1-\rho)$, $\rho < 1$ is the elasticity of substitution.

This form can be derived in the same way as the original form is derived.

Let's start with a solution derived from the differential equation which is derived from the relation between per capita income and wage rate. (Arrow *et al.*, 1961, p.230)

$$y = (c_1 x_1^{\rho} + c_2 x_2^{\rho})^{1/\rho} \tag{3}$$

where c_1, c_2 are two constants of integration. These constants can be determined by the initial (base period) condition. Let us express the constant $c_i, i = 1,2$ by the following two constants $d_i, b_i, i = 1,2$:

$$c_1 = d_1 b_1^{\rho}, c_2 = d_2 b_2^{\rho} \tag{4}$$

The constant $b_i, i = 1,2$ denotes the parameter that can represent factor i's efficiency or the conversion of unit of measurement. The constant $b_i, i = 1,2$ needs to be given exogeneously. The constant $d_i, i = 1,2$ can be expressed by two parameters A, a_1:

$$d_1 + d_2 = A^{\rho}, d_1 = a_1 A^{\rho} \tag{5}$$

Solving for d_2, we obtain $d_2 = (1 - a_1)A^{\rho} = a_2 A^{\rho}$. Thus the production function is written as follows.

$$y = A[a_1(b_1 x_1)^{\rho} + a_2(b_2 x_2)^{\rho}]^{1/\rho} \tag{6}$$

The parameter A denotes scaling constant, the parameter $a_i, i = 1,2$ denotes distribution or cost share parameter, and $b_i, i = 1,2$ plays the role of denoting both the factor i's efficiency and the conversion of unit of measurement. For instance, in the normalized CES form, A corresponds to $\bar{y}$, a_i corresponds to $\bar{\theta}_i$, and b_i corresponds to $1/\bar{x}_i$.

	$\rho \to -\infty$	$\rho \to 0$	$\rho \to 1$	unit cost $p = C(1)$
$A[\sum_{i=1}^2 a_i x_i^{\rho}]^{\frac{1}{\rho}}$ $a_1 + a_2 = 1$	$\min A[x_1, x_2]$	$\Pi_i (A x_i)^{a_i}$	$A \sum_i a_i x_i$	$\frac{1}{A}[\sum a_i^{\sigma} w_i^{1-\sigma}]^{\frac{1}{1-\sigma}}$
$\bar{y}[\sum_{i=1}^2 \bar{\theta}_i (\frac{x_i}{\bar{x}_i})^{\rho}]^{\frac{1}{\rho}}$ $\bar{\theta}_i = \frac{\bar{w}_i \bar{x}_i}{\sum_i \bar{w}_i \bar{x}_i}$	$\min \bar{y}[\frac{x_1}{\bar{x}_1}, \frac{x_2}{\bar{x}_2}]$	$\Pi_i (\bar{y} \frac{x_i}{\bar{x}_i})^{\bar{\theta}_i}$	$\bar{y} \sum_i \bar{\theta}_i \frac{x_i}{\bar{x}_i}$	$\bar{p}[\sum_i \bar{\theta}_i (\frac{w_i}{\bar{w}_i})^{1-\sigma}]^{\frac{1}{1-\sigma}}$ $\bar{p} = [\bar{w}_1 \bar{x}_1 + \bar{w}_2 \bar{x}_2]/\bar{y}$
$A[\sum_{i=1}^2 a_i (b_i x_i)^{\rho}]^{\frac{1}{\rho}}$ $a_1 + a_2 = 1$	$\min A[b_1 x_1, b_2 x_2]$	$\Pi_i (A b_i x_i)^{a_i}$	$A \sum_i a_i b_i x_i$	$\frac{1}{A}[\sum a_i (\frac{w_i}{a_i b_i})^{1-\sigma}]^{\frac{1}{1-\sigma}}$

Table 1: Comparison of the original CES Form, the normalized CES Form, and the canonical CES Form

Note that the unit cost which is the Lagrange multiplier for cost minimization problem is a CES function of effective factor prices with the elasticity of substitution $1/\sigma$. The effective factor i's price $w_i/(a_ib_i)$ is the factor price divided by a_ib_i which represents the efficiency or scale of factor i.

The proposed form has some advantages over the original form of Arrow *et al.* (1961). First, the proposed form includes the normalized form as a special case where $A=\bar{y}, a_i=\bar{w}_i\bar{x}_i/(\bar{w}_1\bar{x}_1+\bar{w}_2\bar{x}_2), b_i=1/\bar{x}_i, i=1,2$. It has the form $A[a_1(bx_1)^\rho+a_2((1-b)x_2)^\rho]^{1/\rho}$ proposed by Barro and Sala-i-Martin (2004) as an alternative form.

Proposition 1: In the canonical CES production function, the parameter a_i is the cost share of factor i if and only if $w_i=sa_ib_i, i=1,2$, where $s>0$ is a constant.

Proof: Consider the following cost minimization problem:

$$\begin{aligned} &C(w_1,w_2,y)=\min_{x_1,x_2} w_1x_1+w_2x_2 \\ &s.t \quad A[a_1(b_1x_1)^{1-\frac{1}{\sigma}}+a_2(b_2x_2)^{1-\frac{1}{\sigma}}]^{\frac{1}{1-\frac{1}{\sigma}}}\geq y \\ &\qquad x_1,x_2\geq 0 \end{aligned} \tag{7}$$

Then the solution is

$$\begin{aligned} x_1&=\frac{1}{Ab_1}(\frac{Ap}{w_1/(a_1b_1)})^\sigma y \\ x_2&=\frac{1}{Ab_2}(\frac{Ap}{w_2/(a_2b_2)})^\sigma y \end{aligned} \tag{8}$$

where $p=\frac{1}{A}[a_1(w_1/(a_1b_1))^{1-\sigma}+a_2(w_2/(a_2b_2))^{1-\sigma}]^{\frac{1}{1-\sigma}}$, which is the Lagrange multiplier and the shadow price of output y.

To find the factor shares at the optimum, we consider the following:

$$\begin{aligned} w_1x_1&=A^{\sigma-1}a_1^\sigma b_1^{\sigma-1}w_1^{1-\sigma}p^\sigma y \\ w_2x_2&=A^{\sigma-1}a_2^\sigma b_2^{\sigma-1}w_2^{1-\sigma}p^\sigma y \\ C(w_1,w_2,y)&=w_1x_1+w_2x_2=py \end{aligned} \tag{9}$$

The cost shares of factors 1 and 2 are

$$\begin{aligned}\frac{w_i x_i}{w_1 x_1 + w_2 x_2} &= \frac{A^{\sigma-1} a_i^{\sigma} b_i^{\sigma-1} w_i^{1-\sigma} p^{\sigma} y}{py} = A^{\sigma-1} a_i \left(\frac{w_i/(a_i b_i)}{p}\right)^{1-\sigma} \\ &= \frac{a_i (w_i/(a_i b_i))^{1-\sigma}}{a_1 (w_1/(a_1 b_1))^{1-\sigma} + a_2 (w_2/(a_2 b_2))^{1-\sigma}}\end{aligned} \tag{10}$$

Since $w_1 = sa_1 b_1, w_2 = sa_2 b_2$ where $s > 0$ is a constant, we obtain

$$\frac{w_1 x_1}{w_1 x_1 + w_2 x_2} = a_1, \frac{w_2 x_2}{w_1 x_1 + w_2 x_2} = a_2 \tag{11}$$

Therefore, the parameters are the cost shares of factors 1 and 2, respectively. Conversely, suppose that $\sigma \neq 1, a_i = \frac{w_i x_i}{w_1 x_1 + w_2 x_2}, i = 1, 2$. Then

$$a_i = \frac{w_i x_i}{w_1 x_1 + w_2 x_2} = \frac{a_i (w_i/(a_i b_i))^{1-\sigma}}{a_1 (w_1/(a_1 b_1))^{1-\sigma} + a_2 (w_2/(a_2 b_2))^{1-\sigma}}$$

Dividing both sides by a_i and multiplying both sides by $a_1 (w_1/(a_1 b_1))^{1-\sigma} + a_2 (w_2/(a_2 b_2))^{1-\sigma}$, we obtain

$$a_1 (w_1/(a_1 b_1))^{1-\sigma} + a_2 (w_2/(a_2 b_2))^{1-\sigma} = (w_i/(a_i b_i))^{1-\sigma}$$

Thus, for $i = 1, a_2 (w_2/(a_2 b_2))^{1-\sigma} = a_2 (w_1/a_1 b_1))^{1-\sigma} \rightarrow \frac{w_2}{a_2 b_2} = \frac{w_1}{a_1 b_1}$. Therefore $w_i = sa_i b_i, i = 1, 2$, where $s > 0$ is a constant. Q.E.D.

We now show that the normalized CES function is a special case of our canonical form. Let us denote the value of the variables in the base period as $\bar{x}_i, \bar{w}_i, \bar{y}, \bar{p}, \bar{C}$.

Proposition 2: If $\bar{w}_i = sa_i b_i, i = 1, 2$, then the canonical CES production function is equal to the normalized CES production function.

Proof: Let $\bar{w}_i = sa_i b_i$. Then $a_i = \frac{\bar{w}_i \bar{x}_i}{\bar{C}}$ and $b_i = \frac{\bar{w}_i}{sa_i} = \frac{\bar{C}}{s\bar{x}_i}$, and it hols that $b_i x_i = \frac{\bar{w}_i x_i}{sa_i} = \frac{\bar{C}}{s\bar{w}_i \bar{x}_i} \bar{w}_i x_i = \frac{1}{s} \bar{p}\bar{y}\frac{x_i}{\bar{x}_i}$. As we have $\bar{p} = s/A, b_i x_i = \frac{\bar{y}}{A}\frac{x_i}{\bar{x}_i}$, substituting it for $b_i x_i$ into the CES production function yields

$$y = A[a_1 (b_1 x_1)^{\rho} + a_2 (b_2 x_2)^{\rho}]^{\frac{1}{\rho}} = \bar{y}[a_1 (\frac{x_1}{\bar{x_1}})^{\rho} + a_2 (\frac{x_2}{\bar{x_2}})^{\rho}]^{\frac{1}{\rho}} \tag{12}$$

This is the normalized (or calibrated share form of) CES production function. Q.E.D.

Second, the proposed form can represent factor-augmenting technological progresses. For instance, suppose that the rate of factor-augmenting technological progress at the period k is denoted by τ_k and the efficiency parameter of factor i in the base year 0 is denoted by $\bar{b_i}$; then, the efficiency parameter in period t is $b_{it} = \Pi_{k=1}^{t}(1+\tau_k)\bar{b_i}$. More generally, we can represent factor-augmenting technological progresses by inserting an additional efficiency term $e_i(t)$ into the term b_{it}: $b_{it} = e_i(t)\bar{b_i}$.

Note that the normalized CES form can also be extended to a dynamic form through the introduction of time dimension. The dynamic form incorporates the factor-augmenting technological progress by inserting the factor efficiency terms $e_i(t)$ into the normalized form as follows:

$$y_t = f(x_1, x_2; t) = \bar{y}[\sum_{i=1}^{2} \bar{\theta}_i(\frac{e_i(t)x_i}{\bar{x_i}})^{\rho}]^{\frac{1}{\rho}} \tag{13}$$

In general, the proposed form can be extended to denote a parametrized family of CES production function. Taking time t as an additional parameter and incorporating the parameter in the coefficient b_i so that it is a function of t, $b_i = b_i(t)$, we obtain a family of CES function indexed by time t,

$$y_t = f(x_1, x_2; t) = A[a_1(b_1(t)x_1)^{\rho} + a_2(b_2(t)x_2)^{\rho}]^{\frac{1}{\rho}} \tag{14}$$

This is different from the production function $Y = [(E_t^K K)^{\frac{\sigma-1}{\sigma}} + (E_t^L L)^{\frac{\sigma-1}{\sigma}}]^{\frac{\sigma}{\sigma-1}}$ in the equation (23) appreared in Klump *et al.* (2012). The constant a_i does not appear in (23) while our form includes the constant $a_i, i = 1,2$ where $a_1 + a_2 = 1, a_1, a_2 \geq 0$. The parameter a_i is related to the factor share of factor i. We can take any parameter q and incorporate it in the coefficient b_i so that it is a function of q, $b_i = b_i(q)$. Then, we obtain a family of CES type functions indexed by q. We call the function the extended CES production function.

Definition 2: The extended CES production function is

$$y_q = f(x_1, x_2; q) = A[a_1(b_1(q)x_1)^{\rho} + a_2(b_2(q)x_2)^{\rho}]^{\frac{1}{\rho}} \tag{15}$$

where A is an efficiency or scale parameter, $a_1 + a_2 = 1, a_1, a_2 \geq 0$, the parameter ρ is a substitution parameter such thta $\rho < 1$.

The parameter q may represent time, nations, regions, or industries.

Third, the proposed form can be regarded as dealing with the issue of dimensional homogeneity more explicitly. Indeed, the parameter b_i can be viewed as including the conversion factor of unit of measurement. The conversion is from the production factor i's unit of measurement to the unit of a certain common dimension. Or the conversion may be into a dimensionless one. In the normalized form, b_i corresponds to $1/\bar{x}_i$ and the conversion is from the factor i's unit into a dimensionless one. The parameter A can be viewed as including the conversion factor from the unit of the common dimension or the dimensionless one into the unit of measurement for output. For instance, in the normalized form, A corresponds to $\bar{y}$ and the conversion is from the dimensionless unit into the unit of measurement for output.

3. CONCLUDING REMARKS

We propose adding an efficiency and conversion term to the original form (Arrow *et al.*, 1961) of the CES production function. The proposed form includes the normalized CES form as a special case. Moreover, it allows us to represent factor-augmenting technological progresses in the context of a dynamic growth model. In general, we can take any parameter q instead of time t and incorporate it in the coefficient b_i so that we obtain a family of CES-type function indexed by q. The parameter q may represent time, nations, regions, or industries. Finally, the proposed form also deals with the concern on dimensional homogeneity well.

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